EVENTUALLY INDEX OF REDUCIBILITY ON SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. It is shown that a module is sequentially Cohen-Macaulay if and only if the index of reducibility for distinguished parameter ideals are eventually constant with special value. As corollaries to the main theorem we given to characterize the Gorensteinness, Cohen-Macaulayness of local rings in term of eventually index of reducibility for distinguished parameter ideals.

1. INTRODUCTION

Throughout this paper let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module of dimension $d > 0$. Then we say that an R-submodule N of M is *irreducible*, if N is not written as the intersection of two larger R-submodules of M. Every R-submodule N of M can be expressed as an irredundant intersection of irreducible R-submodules of M and the number of irreducible R-submodules appearing in such an expression depends only on N and not on the expression. Let us call, for each parameter ideal $\mathfrak{q}$ of M, the number $\mathcal{N}(\mathfrak{q}; M)$ of irreducible R-submodules of M that appear in an irredundant irreducible decomposition of $\mathfrak{q}M$ *the index of reducibility* of M with respect to $\mathfrak{q}$. Let S be the set of parameter ideals of M and a integer number. Then we say that the index of reducibility for S are *eventually* a if there some integer number n such that for all parameter ideals $\mathfrak{q} \in S \cap \mathfrak{m}^n$, we have $\mathcal{N}(\mathfrak{q}; M) = a$.

Firstly, perhaps less widely known is a result of Northcott and Rees which states that if every parameter ideal of R is irreducible then R is Cohen-Macaulay [NR, Theorem 1]. Thus, R is Gorenstein if and only if every parameter ideal is irreducible. Recently, it was shown by many works that the index of reducibility of parameter ideals can be used to deduce a lot of information on the structure of some classes of modules such as Gorenstein rings([No],[NR], [Tr2] [TY]),Cohen-Macaulay modules ([CQT], [Tr], [Tr1], [Tr2], [TY]), Buchsbaum modules ([GSa]), generalized Cohen-Macaulay modules ([CT]), and so on. In [CQT, Theorem 5.2], N. T. Cuong, P. H. Quy and author showed that M is a Cohen-Macaulay module if and only if the index of reducibility for all parameter ideals of M are eventually Cohen-Macaulay type. The necessary condition of this result can be extended by the author in [Tr, Theorem 1.1] for a large class of modules called sequentially Cohen-Macaulay modules.

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To sate this result, let us fix some notation. A filtration

$$\mathcal{D}: 0 \subsetneq D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_\ell = M$$

of R-submodules of M is called the *dimension filtration* of M, if for all $0 \leq i \leq \ell - 1$, D_{i-1} is the largest submodule of D_i with $\dim_R D_{i-1} < \dim_R D_i$, where $\dim_R(0) = -\infty$ for convention. We say that M is *sequentially Cohen-Macaulay*, if $C_i = D_i/D_{i-1}$ is Cohen-Macaulay for all $1 \leq i \leq \ell$. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters of M. Then $\underline{x}$ is said to be *distinguished*, if

$$(x_j \mid d_i < j \leq d)D_i = (0)$$

for all $0 \leq i \leq \ell$, where $d_i = \dim_R D_i$ ([Sch, Definition 2.5]). A parameter ideal $\mathfrak{q}$ of M is called *distinguished*, if there exists a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of M such that $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$. We now denote $r_j(M) = \ell_R((0) :_{\mathrm{H}^j_{\mathfrak{m}}(M)} \mathfrak{m})$ for all $j \in \mathbb{Z}$ and $r(M) = \sum\limits_{j \in \mathbb{Z}} r_j(M)$. Note that $r_d(M)$ is called *Cohen-Macaulay type*. With this notation, the author showed in [Tr, Theorem 1.1] that if M is a sequentially Cohen-Macaulay module, then the index of reducibility for all distinguished parameter ideals of M are eventually $r(M)$. From this point of view, it seems now natural to ask whether the converse of this statement is true. The answer is affirmative, which we are eager to report in the present paper.

Theorem 1.1. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Then the following statements are equivalent.*

(1) *M is sequentially Cohen-Macaulay.*
(2) *The index of reducibility for all distinguished parameter ideals of M are eventually $r(M)$.*
(3) *There exists an integer n such that for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$, we have*

$$\mathcal{N}(\mathfrak{q}; M) \leq r(M).$$

From the main result, we get the following results.

Corollary 1.2. *For all integers n there exists a distinguished parameter ideal $\mathfrak{q} \subseteq \mathfrak{m}^n$, we have*

$$r(M) \leqslant \mathcal{N}(\mathfrak{q}; M)$$

.

Corollary 1.3. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Then the following statements are equivalent.*

(1) *M is Cohen-Macaulay.*
(2) *The index of reducibility for all parameter ideals of M are eventually Cohen-Macaulay type.*
(3) *The index of reducibility for all distinguished parameter ideals of M are eventually Cohen-Macaulay type.*

Theorem 1.4. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Then R is Gorenstein if and only if the index of reducibility for all distinguished parameter ideals are eventually* 1.

Let us explain how this paper is organized. This paper is divided into 3 sections. We shall prove Theorem 1.1 and our corollaries in Section 3. The notion of sequentially Cohen-Macaulay module was introduced by R. Stanley [St] in graded case, and the local case was studied by [Sch]. Our Theorem 1.1 partially covers a main result in [CQT, Theorem 5.2] . In our argument Goto sequence of type II play an important role. In Section 2 let us briefly note the existence of Goto sequence of type II.

2. Goto sequences

Throughout this paper let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module of dimension $d > 0$. We put

$$\operatorname{Assh}_R M = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim R/\mathfrak{p} = d\}.$$

Then

$$\operatorname{Assh}_R M \subseteq \operatorname{Min}_R M \subseteq \operatorname{Ass}_R M.$$

Let $\Lambda(M) = \{\dim_R L \mid L \text{ is an } R\text{-submodule of } M, L \neq (0)\}$. We then have

$$\Lambda(M) = \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M\}.$$

We put $\ell = \sharp\Lambda(M)$ and number the elements $\{d_i\}_{1 \le i \le \ell}$ of $\Lambda(M)$ so that

$$0 \le d_1 < d_2 < \cdots < d_\ell = d.$$

Then because the base ring R is Noetherian, for each $1 \le i \le \ell$ the R-module M contains the largest R-submodule D_i with $\dim_R D_i = d_i$. Therefore, letting $D_0 = (0)$, we have the filtration

$$\mathcal{D} : D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = M$$

of R-submodules of M, which we call the dimension filtration of M. The notion of dimension filtration was firstly given by P. Schenzel [Sch]. Our notion of dimension filtration is a little different from that of [CC, Sch], but throughout this paper let us utilize the above definition. It is standard to check that $\{D_j\}_{0 \le j \le i}$ (resp. $\{D_j/D_i\}_{i \le j \le \ell}$) is the dimension filtration of D_i (resp. M/D_i) for every $1 \le i \le \ell$. We put $C_i = D_i/D_{i-1}$ for $1 \le i \le \ell$.

We note two characterizations of the dimension filtration. Let

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} M(\mathfrak{p})$$

be a primary decomposition of (0) in M, where $M(\mathfrak{p})$ is an R-submodule of M with $\operatorname{Ass}_R M/M(\mathfrak{p}) = \{\mathfrak{p}\}$ for each $\mathfrak{p} \in \operatorname{Ass}_R M$. Then the submodule $D_{\ell-1} = \bigcap_{\mathfrak{p} \in \operatorname{Assh}(M)} M(\mathfrak{p})$ is called the unmixed component of M. We then have the following.

Proposition 2.1 ([Sch, Proposition 2.2, Corollary 2.3])**.** *The following assertions hold true.*

1. $D_i = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M,\ \dim R/\mathfrak{p} \ge d_{i+1}} M(\mathfrak{p})$ *for all* $0 \le i < \ell$.
2. *Let* $1 \le i \le \ell$. *Then* $\operatorname{Ass}_R C_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} = d_i\}$ *and* $\operatorname{Ass}_R D_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} \le d_i\}$.
3. $\operatorname{Ass}_R M/D_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} \ge d_{i+1}\}$ *for all* $1 \le i < \ell$.

We now assume that R is a local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated R-module with $d = \dim_R M \geq 1$ and $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ the dimension filtration. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters of M. Then $\underline{x}$ is said to be *distinguished*, if

$$(x_j \mid d_i < j \leq d)D_i = (0)$$

for all $1 \leq i \leq \ell$, where $d_i = \dim_R D_i$. A parameter ideal $\mathfrak{q}$ of M is called *distinguished*, if there exists a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of M such that $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$. Therefore, if M is a Cohen-Macaulay R-module, every parameter ideal of M is distinguished. Distinguished system of parameters exist and if $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of M, then $x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}$ is also a distinguished system of parameters of M for all integers $n_j \geq 1$.

Settings 2.2. Let $\underline{x} = x_1, x_2, \ldots, x_s$ be a system of elements of R and $\mathfrak{q}_j$ denote the ideal generated by $x_1, \ldots, x_j$ for all $j = 1, \ldots, s$.

Definition 2.3. A system $\underline{x}$ of elements of R is called *Goto sequence* on M, if for all $0 \leq j \leq s-1$ and $0 \leq i \leq \ell$, we have the following

(1) $\operatorname{Ass}(C_i/\mathfrak{q}_j C_i) \subseteq \operatorname{Assh}(C_i/\mathfrak{q}_j C_i) \cup \{\mathfrak{m}\}$,
(2) $x_j D_i = 0$ if $d_i < j \leq d_{i+1}$,
(3) $\mathfrak{q}_{j-1} : x_j = \mathrm{H}^0_{\mathfrak{m}}(M/\mathfrak{q}_{j-1}M)$ and $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/\mathfrak{q}_{j-1}M) - \{\mathfrak{m}\}$.

At first glance, the definition of normal does not seem very intuitive. Once we enter the world of sequences, however, we will see that Goto sequence has a very nice interpretation and properties. We will also see that Goto sequence is useful for many inductive proofs in the next sections. Before we can give some properties of this sequence, we first need reformulate the notion of d-sequences. The sequence $x_1, x_2, \ldots, x_s$ of elements of R is called a d-sequence on M if

$$\mathfrak{q}_i M : x_{i+1} x_j = \mathfrak{q}_i M : x_j$$

for all $0 \leq i < j \leq s$. The concept of a d-sequence is given by Huneke and it plays an important role in the theory of Blow up algebra, e.g. Ress algebra. In the following lemma, we will give some properties of Goto sequences that will be used in the next sections when we study the index of reducibility and the Cohen-Macaulayness of local rings.

Lemma 2.4. *Let $\underline{x} = x_1, x_2, \ldots, x_s$ form a Goto sequence on M. Then we have*

(1) *$\underline{x}$ is part of a system of parameters of M.*
(2) *$\underline{x}$ is a d-sequence.*
(3) *If $d = s$ then $\underline{x}$ is a distinguished system of parameters of M.*

Proof. As an immediate consequence of the definitions we have the first assertion and the thrid assertion. The second assertion is followed from (vii) of [T, Theorem 1.1]. □

Lemma 2.5. *Let R be a homomorphic image of a CohenMacaulay local ring. Assume that system $\underline{x} = x_1, x_2, \ldots, x_d$ of parameters form a Goto sequence on M. Let N denote the unmixed component of $M/\mathfrak{q}_{d-2}M$ and $d \geq 2$. If M/N is Cohen-Macaulay, so is also $M/D_{\ell-1}$.*

Proof. We may assume that $d \geq 3$. Because the assumption of the corollary is inherited to the module $M/\mathfrak{q}_i M$, it is enough to prove the following statement.

Let $x \in R$ be a Goto sequence of length one on M. Let N denote the unmixed component of M/xM and $d \geq 3$. If $\mathrm{H}^i_{\mathfrak{m}}(M/N) = 0$ for $i \leq d-2$ then $\mathrm{H}^i_{\mathfrak{m}}(C_\ell) = 0$ for $i \leq d-1$.

For a submodule N of M, we denote $\overline{N} = (N + xM)/xM$ the submodule of M/xM. Since x is a Goto sequence of length one on M, $\mathrm{Ass}(C_\ell/xC_\ell) \subseteq \mathrm{Assh}(C_\ell/xC_\ell) \cup \{m\}$. Therefore $N/\overline{D}_{\ell-1}$ has a finite length. Since $\overline{M}/N$ is a Cohen-Macaulay module, $H^i_{\mathfrak{m}}(M/D_{\ell-1} + xM) = 0$ for all $0 < i < d-1$. Therefore, we derive from the exact sequence

$$0 \to M/D_{\ell-1} \stackrel{.x}{\to} M/D_{\ell-1} \to M/D_{\ell-1} + xM \to 0$$

the following exact sequence:

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M/D_{\ell-1} + xM) \to \mathrm{H}^1_{\mathfrak{m}}(M/D_{\ell-1}) \stackrel{.x}{\to} \mathrm{H}^1_{\mathfrak{m}}(M/D_{\ell-1}) \to 0.$$

Thus $\mathrm{H}^1_{\mathfrak{m}}(M/D_{\ell-1}) = 0$, and so $N/\overline{D}_{\ell-1} = \mathrm{H}^0_{\mathfrak{m}}(M/D_{\ell-1} + xM) = 0$. Hence $N = \overline{D}_{\ell-1}$. Moreover, since x is $C_\ell = M/D_{\ell-1}$-regular and $C_\ell/xC_\ell \cong \overline{M}/\overline{D}_{\ell-1} = \overline{M}/N$ a Cohen-Macaulay module, C_ℓ is a Cohen-Macaulay module. □

We now denote $r_j(M) = \ell_R((0) :_{\mathrm{H}^j_{\mathfrak{m}}(M)} \mathfrak{m})$ for all $j \in \mathbb{Z}$ and

$$r(M) = \sum_{j \in \mathbb{Z}} r_j(M).$$

Definition 2.6. A system $\underline{x}$ of elements of R is called *Goto sequence of type II* on M, if we have

$$r(M/\mathfrak{q}_j M) \leq r(M/\mathfrak{q}_{j+1} M),$$

for all $0 \leq j \leq s-1$.

Now, we explore the existence of Goto sequence of type II. We have divided the proof of the existence of Goto sequence of type II into sequence of lemmas. First, we begin with the following result of S. Goto and Y. Nakamura [GN].

Lemma 2.7. [GN] *Let R be a homomorphic image of a Cohen-Macaulay local ring and assume that* $\mathrm{Ass}(R) \subseteq \mathrm{Assh}(R) \cup \{\mathfrak{m}\}$. *Then*

$$\mathcal{F} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}_R(\mathfrak{p}) > 1 = \mathrm{depth}(R_\mathfrak{p})\}$$

is a finite set.

The next proposition shows the existence of a special element which is useful for the existence of Goto sequence.

Proposition 2.8. *Let R be a homomorphic image of a Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal of R. Assume that $\mathcal{F} = \{M_i\}_{i=0}^{\ell}$ is a finite filtration of submodules of M such that* $\mathrm{Ass}\, L_i \subseteq \mathrm{Assh}\, L_i \cup \{\mathfrak{m}\}$, *where $L_i = M_i/M_{i-1}$. Then there exists an element $x \in I$ satisfies the following conditions*

1. $\mathrm{Ass}(L_i/x^n L_i) \subseteq \mathrm{Assh}(L_i/x^n L_i) \cup \{\mathfrak{m}\}$, *For all $i = 0, \ldots, \ell-1$.*
2. $x \notin \mathfrak{p}$, *for all* $\mathfrak{p} \in \mathrm{Ass}(M) - \{\mathfrak{m}\}$.
3. $(0) :_{L_i} x = \mathrm{H}^0_{\mathfrak{m}}(L_i)$ *and* $(0) :_M x = \mathrm{H}^0_{\mathfrak{m}}(M)$, *for all $i = 0, \ldots, \ell-1$,*

Proof. Set $I_i = \mathrm{Ann}(L_i)$, and $R_i = R/I_i$, then $\mathrm{Ass}(R_i) \subseteq \mathrm{Assh}(R_i) \cup \{\mathfrak{m}\}$ and $\dim R/I_i > \dim R/I_{i+1}$ for all $i = 0, \ldots, s-1$. Moreover, we have

$\mathrm{Ass}(R_i) = \mathrm{Ass}(L_i) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \in \mathrm{Ass}(M) \text{ and } \dim R/\mathfrak{p} = \dim R/I_i = d_i\} \cup \{\mathfrak{m}\}$.

Set

$$\mathcal{F}_i = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I_i \subset \mathfrak{p} \text{ and } \mathrm{ht}_{R_i}(\mathfrak{p}/I_i) > 1 = \mathrm{depth}((L_i)_{\mathfrak{p}})\}.$$

By Lemma 2.7 and the fact $\mathrm{Ass}(L_i) \subseteq \mathrm{Assh}(L_i) \cup \{\mathfrak{m}\}$, we see that the set

$$\{\mathfrak{p} \in \mathrm{Spec}(R_i) \mid \mathrm{ht}_{R_i}(\mathfrak{p}) > 1 = \mathrm{depth}((L_i)_{\mathfrak{p}})\}$$

is finite, and so that $\mathcal{F}_i$ are a finite set for all $i = 1, \ldots, \ell$. Put $\mathcal{F} = \mathrm{Ass}(M) \cup \bigcup_{i=1}^{t} \mathcal{F}_i \setminus \{\mathfrak{m}\}$. By the Prime Avoidance Theorem, we can choose $y \in I$ such that $y \notin \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}$ and $\dim M_i/yM_i = \dim M_i - 1$ for all $i = 1, \ldots, \ell$. On the other hand, we can choose an integer n_0 such that $(0) :_M y^n = (0) :_M y^{n_0}$ and $(0) :_{L_i} y^n = (0) :_{L_i} y^{n_0}$, for all $n \geq n_0$ and $i = 1, \ldots, \ell$. Put $x = y^{n_0+1}$. Then we have $x \notin \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}$ and $(0) :_{L_i} x^2 = (0) :_{L_i} x$ for all $i = 1, \ldots, \ell$. Now we show that x have the conditions as required.

First let us prove the condition (1). To this end, consider $\mathfrak{p} \in \mathrm{Ass}(N_i/xN_i)$ with $\mathfrak{p} \neq \mathfrak{m}$. Then we have $\mathrm{depth}(L_i/xL_i)_{\mathfrak{p}} = 0$. On the other hand, $\mathrm{depth}(M_i)_{\mathfrak{p}} > 0$ since $\mathfrak{p} \notin \mathrm{Ass}(L_i) \subseteq \mathrm{Ass}(M)$. Hence $\mathrm{depth}(L_i)_{\mathfrak{p}} = 1$. It implies that $\mathrm{ht}_{R_i}(\mathfrak{p}) = 1$, since $\mathfrak{p} \notin \mathcal{F}_i$. By the assumption R_i is a catenary ring, therefore

$$\dim R/\mathfrak{p} = \dim R_i - \mathrm{ht}_{R_i}(\mathfrak{p}) = \dim R_i/xR_i = \dim L_i/xL_i.$$

Hence $\mathfrak{p} \in \mathrm{Assh}(L_i/xL_i)$.

Since the condition (2) is trivial, it remains to prove the condition (3). Take $\mathfrak{p} \in \mathrm{Ass}_R(0) :_{L_i} x$ with $\mathfrak{p} \neq \mathfrak{m}$, Then $x \notin \mathfrak{p}$ as $\mathfrak{p} \in \mathrm{Assh}(L_i)$. Hence $((0) :_{L_i} x)_{\mathfrak{p}} = (0)$ and this is a contradiction. It implies that $(0) :_{N_i} x$ is finite length. Since $(0) :_{L_i} x^2 = (0) :_{L_i} x$, we have $(0) :_{L_i} x = \mathrm{H}^0_{\mathfrak{m}}(L_i)$. It follows from the following exact sequence

$$0 \to M_{i-1} \to M_i \to L_i \to 0$$

and $x \, \mathrm{H}^0_{\mathfrak{m}}(L_i) = 0$ for all $i = 1, \ldots, \ell$ that the following sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M_{i-1}) \to \mathrm{H}^0_{\mathfrak{m}}(M_i) \to \mathrm{H}^0_{\mathfrak{m}}(L_i)$$

$$\text{and } 0 \to (0) :_{M_{i-1}} x \to (0) :_{M_i} x \to (0) :_{L_i} x$$

are exact. By induction and $(0) :_M x = (0) :_M x^2$, we have $(0) :_M x = \mathrm{H}^0_{\mathfrak{m}}(M)$ and this completes the proof. □

The existence of Goto sequence is established by our next Corollary.

Corollary 2.9. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal of R. Then there exists a system $\underline{x} = x_1, x_2, \ldots, x_s$ of elements of I such that $\underline{x}$ is a Goto sequence on M.*

Proof. We prove this by induction on s, the case in which $s = 1$ having been dealt with in Lemma 2.8. So we suppose that $s = j \geq 2$ and that the result has been proved for smaller values of s. Suppose that $d_i < j \leq d_{i+1}$ for some i. We see immediately from this induction hypothesis that

$$\mathrm{Ass}(N_i/\mathfrak{q}_{j-1}N_i) \subseteq \mathrm{Assh}(N_{j-1}/\mathfrak{q}_{j-1}N_i) \cup \{\mathfrak{m}\},$$

where $\mathfrak{q}_{j-1} = (x_1, \dots, x_{j-1})$, for all $i = 0, \dots, \ell-1$. Moreover the sequence $x_1, x_2, \dots, x_{d_i}$ is a system of parameters of D_i. Therefore $\operatorname{Ann}(D_i) + \mathfrak{q}_{j-1}$ is $\mathfrak{m}$-primary ideals. So that, by Lemma 2.8, there exists an element $x_j \in I \cap \operatorname{Ann}(D_i)$, as required. This completes the inductive step, and the proof. $\square$

Let $\mathfrak{q} = (x_1, x_2, \dots, x_d)$ be a parameter ideal in R and let M be an R-module. For each integer $n \geq 1$ we denote by $\underline{x}^n$ the sequence $x_1^n, x_2^n, \dots, x_d^n$. Let $K^\bullet(x^n)$ be the Koszul complex of R generated by the sequence $\underline{x}^n$ and let $H^\bullet(\underline{x}^n; M) = H^\bullet(\operatorname{Hom}_R(K^\bullet(\underline{x}^n), M))$ be the Koszul cohomology module of M. Then for every $p \in \mathbb{Z}$ the family $\{H^p(\underline{x}^n; M)\}_{n \geq 1}$ naturally forms an inductive system of R-modules, whose limit

$$H^p_{\mathfrak{q}} = \lim_{n \to \infty} H^p(\underline{x}^n; M)$$

is isomorphic to the local cohomology module

$$H^p_{\mathfrak{m}}(M) = \lim_{n \to \infty} \operatorname{Ext}^p_R(R/\mathfrak{m}^n, M)$$

For each $n \geq 1$ and $p \in \mathbb{Z}$ let $\phi^{p,n}_{\underline{x},M} : H^p(\underline{x}^n; M) \to H^p_{\mathfrak{m}}(M)$ denote the canonical homomorphism into the limit.

With this notation we have the following result.

Lemma 2.10 ([GS1], Lemma 1.7)**.** *Let M be a finitely generated R-module and x an M-regular element and $\underline{x} = x_1, \dots, x_r$ be a system of elements in R with $x_1 = x$. Then there exists a splitting exact sequence for each $p \in \mathbb{Z}$,*

$$0 \to H^p(\underline{x}; M) \to H^p(\underline{x}; M/xM) \to H^{p+1}(\underline{x}; M) \to 0.$$

Definition 2.11 ([GSa] Lemma 3.12)**.** Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim R = d \geq 1$. Let M be a finitely generated R-module. Then there exists an integer n_0 such that for all systems of parameters $\underline{x} = x_1, \dots, x_d$ for R contained in $\mathfrak{m}^{n_0}$ and for all $p \in \mathbb{Z}$, the canonical homomorphisms

$$\phi^{p,1}_{\underline{x},M} : H^p(\underline{x}, M) \to H^p_{\mathfrak{m}}(M)$$

into the inductive limit are surjective on the socles. The least integer n_0 with this property is called a Goto number of R-module M and denote by $\mathrm{g}(M)$.

We need the following result in next section.

Lemma 2.12. *Assume that N is a submodule of M such that M/N is Cohen-Macaulay and $\dim N < \dim M$. Then for all parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^{\mathrm{g}(M)}$ of M, we have*

$$\mathcal{N}(\mathfrak{q}; M) = \mathcal{N}(\mathfrak{q}; N) + \mathcal{N}(\mathfrak{q}; M/N).$$

Proof. Let $\mathfrak{q} = (x_1, x_2 \dots, x_d)$ be a parameter ideal of M such that $\mathfrak{q} \subseteq \mathfrak{m}^n$. Then by the definition of Goto number, the canonical map

$$\phi_M : M/(x_1, x_2, \dots, x_d)M \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(M) = \lim_{q \to \infty} M/(x_1^q, x_2^q, \dots, x_d^q)M$$

is surjective on the socles. We put $\mathcal{M} = M/N$ and look at the exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\epsilon} \mathcal{M} \longrightarrow 0$$

of R-modules, where ι (resp. ϵ) denotes the embedding (resp. the canonical epimorphism). Then, since $\dim M > \dim_R N$ and since $x_1, x_2, \ldots, x_d$ is a regular sequence for $\mathcal{M}$, we get the following commutative diagram

$$\begin{CD}
@. @. @. 0 @.\\
@. @. @. @VVV @.\\
0 @>>> N/\mathfrak{q}N @>{\overline{\iota}}>> M/\mathfrak{q}M @>{\overline{\epsilon}}>> \mathcal{M}/\mathfrak{q}\mathcal{M} @>>> 0\\
@. @. @VV{\phi_M}V @VV{\phi_{\mathcal{M}}}V @.\\
@. @. \mathrm{H}^d_{\mathfrak{m}}(M) @>{=}>> \mathrm{H}^d_{\mathfrak{m}}(\mathcal{M}) @.
\end{CD}$$

with exact first row. Let $x \in (0) :_{\mathcal{M}/\mathfrak{q}\mathcal{M}} \mathfrak{m}$. Then, since ϕ_M is surjective on the socles, we get an element $y \in (0) :_{M/\mathfrak{q}M} \mathfrak{m}$ such that $\phi_{\mathcal{M}}(x) = \phi_M(y)$. Thus $\overline{\epsilon}(y) = x$, because the canonical map $\phi_{\mathcal{M}}$ is injective, whence

$$[N + \mathfrak{q}M] :_M \mathfrak{m} = N + [\mathfrak{q}M :_M \mathfrak{m}].$$

Moreover we have

$$\mathcal{N}(\mathfrak{q}; M) = \mathcal{N}(\mathfrak{q}; N) + \mathcal{N}(\mathfrak{q}; M/N).$$

$\square$

Lemma 2.13. *Let M be a finitely generated R-module. Assume that x is an M-regular element of M such that $x \in \mathfrak{m}^{\mathrm{g}(M)}$. Then we have*

$$r_i(M) \leq r_{i-1}(M/xM)$$

for all $i \in \mathbb{Z}$

Proof. Let $x_2, \ldots, x_d$ be a system of parameters of module M/xM such that $x_i \in \mathfrak{m}^{\mathrm{g}(M)}$. Put $\underline{x} = x_1, x_2, \ldots, x_d$ and $\mathfrak{q} = (\underline{x})$, where $x_1 = x$. Since $x \in \mathfrak{m}^{\mathrm{g}(M)}$, we have $\mathfrak{q} \subseteq \mathfrak{m}^{\mathrm{g}(M)}$. By the definition of Goto number, we have the canonical homomorphism

$$H^i(\underline{x}, M) \to H^i_{\mathfrak{m}}(M)$$

into the inductive limit are surjective on the socles, for each $i \in \mathbb{Z}$. By the regularity of $x = x_1$ on M, it follows from the following sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow M/xM \longrightarrow 0$$

that there are induced the diagram

$$\begin{CD}
0 @>>> H^i(\underline{x}; M) @>>> H^i(\underline{x}, M/xM) @>>> H^{i+1}(\underline{x}; M) @>>> 0\\
@. @VVV @VVV @VVV @.\\
@>>> H^i_{\mathfrak{m}}(M) @>>> H^i_{\mathfrak{m}}(M/xM) @>>> H^{i+1}_{\mathfrak{m}}(M) @>>>
\end{CD}$$

commutes, for all $i \in \mathbb{Z}$. It follows from the above commutative diagrams and Lemma 2.10 that after applying the functor $\mathrm{Hom}(k, *)$, we obtain the commutative diagram

$$\begin{CD}
\mathrm{Hom}(k, H^i(\underline{x}, M/xM)) @>>> \mathrm{Hom}(k, H^{i+1}(\underline{x}; M)) @>>> 0\\
@VVV @VVV @.\\
\mathrm{Hom}(k, H^i_{\mathfrak{m}}(M/xM)) @>>> \mathrm{Hom}(k, H^{i+1}_{\mathfrak{m}}(M)) @.
\end{CD}$$

for all $i \in \mathbb{Z}$. Since the map $\operatorname{Hom}(k, H^{i+1}(\underline{x}; M)) \to \operatorname{Hom}(k, H^{i+1}_{\mathfrak{m}}(M))$ is surjective, so is the map $\operatorname{Hom}(k, H^i_{\mathfrak{m}}(M/xM)) \to \operatorname{Hom}(k, H^{i+1}_{\mathfrak{m}}(M))$. Therefore the map $\operatorname{Hom}(k, H^i(\underline{x}; M/xM)) \to \operatorname{Hom}(k, H^i_{\mathfrak{m}}(M/xM))$ is surjective and $r_i(M) \le r_{i-1}(M/xM)$ for all $i \in \mathbb{Z}$. This completes the proof.

□

Lemma 2.14. *Let M be a finitely generated R-module with $\dim M = s \ge 1$. Let k and l be two positive integers. Then there exists an integer $n_3 > l$ such that*

$$(\mathfrak{m}^{n_3} + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m}^k \subseteq \mathfrak{m}^l M + \mathrm{H}^0_{\mathfrak{m}}(M).$$

Proof. Let $\overline{M} = M/\mathrm{H}^0_{\mathfrak{m}}(M)$. Then there is an $\overline{M}$-regular element a contained in $\mathfrak{m}^k$. By the Artin-Rees Lemma, there exists a positive integer m such that $\mathfrak{m}^{l+m}\overline{M} \cap a\overline{M} = \mathfrak{m}^l(\mathfrak{m}^m\overline{M} \cap a\overline{M})$. Set $n_3 = l + m$. We have

$$a(\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k) \subseteq a(\mathfrak{m}^{n_3}\overline{M} : a) = \mathfrak{m}^{n_3}\overline{M} \cap a\overline{M} = \mathfrak{m}^l(\mathfrak{m}^m\overline{M} \cap a\overline{M}),$$

and so $a(\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k) \subseteq a\mathfrak{m}^l\overline{M}$. It follows from the regularity of a that $\mathfrak{m}^{n_3}\overline{M} : \mathfrak{m}^k \subseteq \mathfrak{m}^l\overline{M}$. Hence $(\mathfrak{m}^{n_3}M + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m}^k \subseteq \mathfrak{m}^l M + \mathrm{H}^0_{\mathfrak{m}}(M)$, as required. □

Lemma 2.15. *Let M be a finitely generated R-module with $\dim M = s \ge 1$. Then there exists a positive integer n_4 such that for all $x \in \mathfrak{m}^{n_4}$, we have*

$$r_0(M) + r_1(M) \le r_0(M/xM).$$

Proof. Since $H^0_{\mathfrak{m}}(M)$ have finite length, there exists an integer l such that $\mathfrak{m}^l M \cap \mathrm{H}^0_{\mathfrak{m}}(M) = 0$. By Lemma 2.14, there is an integer $n_4 > \max\{l, \mathrm{g}(M/\mathrm{H}^0_{\mathfrak{m}}(M))\}$ such that for all $x \in \mathfrak{m}^{n_4}$ we have

$$(xM + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m} \subseteq (\mathfrak{m}^{n_4}M + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m} \subseteq \mathfrak{m}^l M + \mathrm{H}^0_{\mathfrak{m}}(M).$$

Let $b \in (xM + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m}$. Write $b = \alpha + \beta$ with $\alpha \in \mathfrak{m}^l M$ and $\beta \in \mathrm{H}^0_{\mathfrak{m}}(M)$. Then, since $x \in \mathfrak{m}^{n_4} \subseteq \mathfrak{m}^{l+1}$, we get that

$$\mathfrak{m}\alpha \subseteq \mathfrak{m}^{l+1}M \cap (xM + \mathrm{H}^0_{\mathfrak{m}}(M)) = xM + \mathfrak{m}^{l+1}M \cap \mathrm{H}^0_{\mathfrak{m}}(M) = xM.$$

Thus $\alpha \in xM : \mathfrak{m}$, and so $(xM + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m} = xM : \mathfrak{m} + \mathrm{H}^0_{\mathfrak{m}}(M)$. Since $\mathfrak{m}^l M \cap H^0_{\mathfrak{m}}(M) = 0$, we have the following exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \to M/xM \to M/(xM + \mathrm{H}^0_{\mathfrak{m}}(M)) \to 0.$$

It follow that the sequence

$$0 \to (0) :_M \mathfrak{m} \to (xM : \mathfrak{m})/xM \to ((xM + \mathrm{H}^0_{\mathfrak{m}}(M)) : \mathfrak{m})/(xM + \mathrm{H}^0_{\mathfrak{m}}(M)) \to 0$$

is exact. Therefore, for all $x \in \mathfrak{m}^{n_4}$ we have

$$r_0(M) + r_0(M/xM + \mathrm{H}^0_{\mathfrak{m}}(M)) \le r_0(M/xM).$$

Since $n_4 \ge \mathrm{g}(M/\mathrm{H}^0_{\mathfrak{m}}(M))$, by Lemma 2.13, we have $r_1(M/H^0_{\mathfrak{m}}(M)) \le r_0(M/xM + \mathrm{H}^0_{\mathfrak{m}}(M))$. Since $r_1(M) = r_1(M/H^0_{\mathfrak{m}}(M))$, we have

$$r_0(M) + r_1(M) \le r_0(M/xM),$$

as required.

□

Corollary 2.16. *Let M be a finitely generated R-module with $\dim M \geq 2$. Then there exists an integer n such that for all parameter elements $x \in \mathfrak{m}^n$, we have*

$$r(M) \leq r(M/xM).$$

Proof. Choose n_1, as lager as possible, such that $n_1 \geq \mathrm{g}(M/\operatorname{H}^0_{\mathfrak{m}}(M))$ and for all parameter $x \in \mathfrak{m}^{n_1}$, we have

$$r_0(M) + r_1(M) \leq r_0(M/xM).$$

Since $\operatorname{H}^i_{\mathfrak{m}}(M) = \operatorname{H}^i_{\mathfrak{m}}(M/\operatorname{H}^0_{\mathfrak{m}}(M))$ for all $i \geq 1$, by Lemma 2.13, we have $r_i(M) \leq r_{i-1}(M/xM)$ for all $i \geq 2$. Hence for all parameter $x \in \mathfrak{m}^{n_1}$, we have

$$r(M) \leq r(M/xM),$$

as required. $\square$

The existence of Goto sequence of type II is established by our next Proposition.

Proposition 2.17. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal of R. Then there exists a system $\underline{x} = x_1, x_2, \ldots, x_s$ of elements of I such that $\underline{x}$ is a Goto sequence of type II on M.*

Proof. We shall now show the our result by induction on s. In the case in which $s = 1$ there is nothing to prove, because of the Lemma 2.8. So assume inductively that $j \in \mathbb{N}$ with $j > 1$ and and that the desired result has been established when $s = j-1$. Suppose that $d_i < j \leq d_{i+1}$ for some i. By induction we have system $x_1, \ldots, x_{j-1}$ of R such that satisfies the following conditions

(1) $\operatorname{Ass}(N_i/\mathfrak{q}_{j-1}N_i) \subseteq \operatorname{Assh}(N_j/\mathfrak{q}_{j-1}N_i) \cup \{\mathfrak{m}\}$, where $\mathfrak{q}_{j-1} = (x_1, \ldots, x_{j-1})$, for all $i = 0, \ldots, \ell - 1$.
(2) The sequence $x_1, x_2, \ldots, x_{d_i}$ is a system of parameters of D_i.

Let $\overline{R} = R/\mathfrak{q}_{j-1}$ $\overline{M} = M/\mathfrak{q}_{j-1}M$ $\mathfrak{n} = \mathfrak{m}/\mathfrak{q}_{j-1}$.

Choose n, as lager as possible, such that for all parameter $x \in \mathfrak{m}^n$, we have

$$r(\overline{M}) \leq r(\overline{M}/x\overline{M}).$$

Put $J = (\operatorname{Ann}(D_i) + \mathfrak{q}_{j-1}) \cap I \cap \mathfrak{m}^n$. Then $J\overline{R}$ is an $\mathfrak{n}$-primary ideal of $\overline{R}$. By Lemma 2.8, we can choose $x_{j+1} \in \operatorname{Ann}(D_i) \cap I \cap \mathfrak{m}^n$, as required. With this observation, we can complete the inductive step and the proof. $\square$

3. The Proof of Main theorem

The purpose of this section is to prove the equivalence of assertions (1) and (2) in Theorem 1.1. We maintain the following settings.

Settings 3.1. Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Let $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of M with $\dim D_i = d_i$. We put $L = M/D_{\ell-1}$.

The notion of a sequentially Cohen-Macaulay module was introduced first by Stanley [St] for the graded case and in [Sch] for the local case.

Definition 3.2 ([Sch, St]). We say that M is a *sequentially Cohen-Macaulay R-module*, if C_i is a Cohen-Macaulay R-module for all $1 \leq i \leq \ell$, where $C_i = D_i/D_{i-1}$.

Recall that an R-submodule N of M is *irreducible*, if N is not written as the intersection of two larger R-submodules of M. Every R-submodule N of M can be expressed as an irredundant intersection of irreducible R-submodules of M and the number of irreducible R-submodules appearing in such an expression depends only on N and not on the expression. Let us call, for each parameter ideal $\mathfrak{q}$ of M, the number $\mathcal{N}(\mathfrak{q}; M)$ of irreducible R-submodules of M that appear in an irredundant irreducible decomposition of $\mathfrak{q}M$ the index of reducibility of M with respect to $\mathfrak{q}$. Let S be the set of parameter ideals of M and a integer number. Then we say that the index of reducibility for S are *eventually* a if there some integer number n such that for all parameter ideals $\mathfrak{q} \in S \cap \mathfrak{m}^n$, we have

$$\mathcal{N}(\mathfrak{q}; M) = a.$$

We then have the following.

Theorem 3.3 ([Tr, Theorem 1.1]). *Suppose that M is a sequentially Cohen-Macaulay R-module. Then the index of reducibility for all distinguished parameter ideals of M are eventually $r(M)$.*

Our next major aim is the establishment of a proof of converse of the above Theorem. To prepare the ground for this, we begin the following result.

Proposition 3.4. *Assume that $d \geq 2$ and the index of reducibility for all distinguished parameter ideals of M are eventually $r(M)$. Then L is Cohen-Macaulay.*

Proof. Since the index of reducibility for all distinguished parameter ideals of M are eventually $r(M)$, there exists an integer n such that for all distinguish parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$ we have

$$\mathcal{N}(\mathfrak{q}; M) = r(M).$$

By Proposition 2.17, there exists a Goto sequence $x_1, \ldots, x_{d-2}$ of type II in $\mathfrak{m}^n$. Let $\mathfrak{q}_{d-2} = (x_1, \ldots, x_{d-2})$ and $A = M/\mathfrak{q}_{d-2}M$ and let N denote the unmixed component of A. Then A/N is a generalized Cohen-Macaulay R-module since $\dim A/N = 2$ and A/N is unmixed. Therefore there exists an integer $n_0 > n$ such that for all parameters $x \in \mathfrak{m}^{n_0}$, we have $xH^1_{\mathfrak{m}}(A/N) = 0$, $r_1(A/(x)+N) = r_1(A/N)+r_2(A/N)$ and $\mathcal{N}(x; N) = r_0(N)+r_1(N)$, because $\dim N \leq 1$. Suppose that $d_{i_0} < d-2 \leq d_{i_0+1}$ for some i_0. Then $\operatorname{Ann}(\mathfrak{a}_{i_0}) + \mathfrak{q}_{d-2}$ is an $\mathfrak{m}$-primary ideal of R. Then we can choose $x_{d-1} \in \mathfrak{m}^{n_0} \cap \operatorname{Ann}(\mathfrak{a}_{i_0})$ as in Proposition 2.8.

Let $\mathfrak{q}_{d-1} = (\mathfrak{q}_{d-2}, x_{d-1})$ and $B = M/\mathfrak{q}_{d-1}M$. Since $\dim B = 1$, B is sequentially Cohen-Macaulay. It follows from $\operatorname{Ann} \mathfrak{a}_{\ell-1} + \mathfrak{q}_{d-1}$ is an $\mathfrak{m}$-primary ideal of R and Theorem 3.3, we have choose $x_d \in \operatorname{Ann} \mathfrak{a}_{\ell-1}$ such that

$$\mathcal{N}(x_dB; B) = r_1(B) + r_0(B).$$

On the other hand, it follows from the exact sequence

$$0 \to N \to A \to A/N \to 0$$

and x_{d-1} is a regular on A/N that $r_2(A/N) = r_2(A)$ and

$$0 \to N/x_{d-1}N \to B \to A/(x_{d-1}) + N \to 0.$$

Since $\dim N/x_{d-1}N = 0$, the sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(N/x_{d-1}N) \to \mathrm{H}^0_{\mathfrak{m}}(B) \to \mathrm{H}^0_{\mathfrak{m}}(A/(x_{d-1}) + N) \to 0.$$

is exact and $\mathrm{H}^1_{\mathfrak{m}}(B) = \mathrm{H}^1_{\mathfrak{m}}(A/(x_{d-1}) + N)$. Thus

$$r_1(B) = r_1(A/(x_{d-1}) + N) = r_1(A/N) + r_2(A/N) = r_1(A/N) + r_2(A),$$

because of the choice of x_{d-1}. It follows from the above exact sequence that the sequence

$$0 \to (0) :_{N/x_{d-1}N} \mathfrak{m} \to (0) :_{\mathrm{H}^0_{\mathfrak{m}}(B)} \mathfrak{m} \to (0) :_{\mathrm{H}^0_{\mathfrak{m}}(A/x_{d-1}A+N)} \mathfrak{m}.$$

is exact and so $r_0(N) + r_1(N) = \mathcal{N}(x_{d-1}; N) \leq r_0(B)$, because of the choice of x_{d-1}. Since $r_0(B) + r_1(B) = \mathcal{N}(x_d; B)$, therefore we have

$$r_0(N) + r_1(N) + r_1(A/N) + r_2(A) \leq \mathcal{N}(x_d; B) = \mathcal{N}(\mathfrak{q}; M)$$

By the definition of Goto sequence of type II, we have $\mathcal{N}(\mathfrak{q}; M) \leq r(M) \leq r(A)$. Thus $r_1(N) + r_1(A/N) \leq r_1(A)$ because $r_0(N) = r_0(A)$ and $r(A) = r_0(A) + r_1(A) + r_2(A)$. On the other hand, it follows from the exact sequence

$$0 \to N \to A \to A/N \to 0$$

that the sequence

$$0 \to \mathrm{H}^1_{\mathfrak{m}}(N) \to \mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(A/N) \to 0$$

is exact. Thus $r_1(A) \leq r_1(N) + r_1(A/N)$ and so $r_1(N) + r_1(A/N) = r_1(A)$. It follows that

$$r_0(N) + r_1(N) + r_1(A/N) + r_2(A) = \mathcal{N}(x_d; B).$$

Since $r_0(B) + r_1(B) = \mathcal{N}(x_d; B)$ and $r_1(B) = r_1(A/N) + r_2(A)$, therefore we have $\mathcal{N}(x_{d-1}; N) = r_0(B)$. It follows from the above exact sequence that $\mathrm{H}^0_{\mathfrak{m}}(A/x_{d-1}A+N) = 0$. Now we derive from the exact sequence

$$0 \to A/N \overset{.x_{d-1}}{\to} A/N \to A/N + x_{d-1}A \to 0$$

the following exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(A/N + x_{d-1}A) \to \mathrm{H}^1_{\mathfrak{m}}(A/N) \overset{.x_{d-1}}{\to} \mathrm{H}^1_{\mathfrak{m}}(A/N).$$

Thus $\mathrm{H}^1_{\mathfrak{m}}(A/N) = 0$, because $x_{d-1}\,\mathrm{H}^1_{\mathfrak{m}}(M) = 0$. Hence L is Cohen-Macaulay, because of Lemma 2.5, and the proof is complete. □

The next result is of special significance as it is used as a basis for proving the Theorem 1.1, and is formulated as follows.

Proposition 3.5. *Assume that there exists an integer n such that for all distinguish parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$ we have*

$$\mathcal{N}(\mathfrak{q}; M) \leq r(M).$$

Then R is sequentially Cohen-Macaulay.

Proof. We use induction on the dimensional d of M. In the case in which $\dim M = 1$, it is clear that R is sequentially Cohen-Macaulay. Now suppose, inductively, that $d > 1$ and that the result has been proved for smaller values of d. Recall that $D_{\ell-1}$ is the unmixed component of M. Therefore, by the Prime Avoidance Theorem, we can choose the part of a system $x_{d_{\ell-1}+1}, \ldots, x_d$ of parameters of R such that $\mathfrak{b} \subseteq \mathfrak{m}^n$ and $\mathfrak{b}M \cap D_{\ell-1} = 0$, where $\mathfrak{b} = (x_{d_{\ell-1}+1}, \ldots, x_d)$. On the other hand, since $\mathcal{N}(\mathfrak{q}; M) \leq r(M)$ for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$, by Proposition 3.4, L is Cohen-Macaulay.

A simple inductive argument therefore shows that it is enough for us to prove that there exists n_1 such that for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^{n_1}$ of $D_{\ell-1}$, we have

$$\mathcal{N}(\mathfrak{q}; D_{\ell-1}) = r(D_{\ell-1}).$$

In fact, since L is Cohen-Macaulay, straightforwardly by Lemma 2.12, there exists n_0 such that for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^{n_0}$, we have

$$\mathcal{N}(\mathfrak{q}; M) = \mathcal{N}(\mathfrak{q}; D_{\ell-1}) + \mathcal{N}(\mathfrak{q}; L).$$

Choose $n_1 \geq \max\{n_0, n\}$. Assume that $x_1, \ldots, x_{d_{\ell-1}}$ is a distinguished system of parameters of $D_{\ell-1}$ such that $(x_1, \ldots, x_{d_{\ell-1}}) \subseteq \mathfrak{m}^{n_1}$. It would then follow from the definition of distinguished system of parameter that system $x_1, \ldots, x_d$ of parameters is distinguished of M such that $\mathfrak{q} \subseteq \mathfrak{m}^{n_1}$, where $\mathfrak{q} = (x_1, \ldots, x_{d_{\ell-1}}, \mathfrak{b})$. Since $n_1 \geq n$, we have $\mathcal{N}(\mathfrak{q}; M) = r(M) = \sum\limits_{j \in \mathbb{Z}} r_j(M)$. On other hand, it follows from L is Cohen-Macaulay and the exact sequence

$$0 \to D_{\ell-1} \to M \to L \to 0$$

that the following sequence

$$0 \to \mathrm{H}^d_{\mathfrak{m}}(D_{\ell-1}) \to \mathrm{H}^d_{\mathfrak{m}}(M) \to \mathrm{H}^d_{\mathfrak{m}}(L) \to 0$$

is exact. Since $\dim D_{\ell-1} < d$, we have $\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(L)$ and so $r_d(M) = r_d(L)$. Since L is Cohen-Macaulay and $\mathfrak{q}$ is a parameter ideal of L, we have $\mathcal{N}(\mathfrak{q}; L) = r_d(L)$. Therefore, it follows that we obtain $\mathcal{N}(\mathfrak{q}; D_{\ell-1}) = r(D_{\ell-1})$. Hence for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$ of $D_{\ell-1}$, we have

$$\mathcal{N}(\mathfrak{q}; D_{\ell-1}) = r(D_{\ell-1}).$$

□

Proof of Theorem 1.1. (1) ⇒ (2) This is now immediate from Theorem 3.3.
(2) ⇒ (3) is trivial.
(3) ⇒ (1) This now immediate from Proposition 3.5.

□

Corollary 3.6. *The following statements are equivalent.*

1. *M is Cohen-Macaulay.*
2. *The index of reducibility for all parameter ideals of M are eventually $r_d(M)$.*
3. *The index of reducibility for all distinguished parameter ideals of M are eventually $r_d(M)$.*
4. *There exists an integer n such that for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$, we have*

$$\mathcal{N}(\mathfrak{q}; M) \leq r_d(M).$$

Proof. (1) ⇒ (2) This is now immediate from Theorem 1.1.
(2) ⇒ (3) and (3) ⇒ (4) are trivial.
(4) ⇒ (1) Since $r_d(M) \leq \sum\limits_{j \in \mathbb{Z}} r_j(M) = r(M)$, for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$, we have $\mathcal{N}(\mathfrak{q}; M) \leq r(M)$. Thus by Theorem 1.1, M is sequentially Cohen-Macaulay. By Theorem 3.3, there exists a distinguished parameter ideal $\mathfrak{q} \subseteq$ such that we have $r(M) = \mathcal{N}(\mathfrak{q}; M)$. It follows from the hypothesis that $r(M) = r_d(M)$, and so $r_i(M) = 0$ for all $i < d$. Hence M is Cohen-Macaulay.

□

Theorem 3.7. *R is Gorenstein if and only if the index of reducibility for all distinguished parameter ideals are eventually* 1.

Proof. Only if: Since R is Gorenstein, R is Cohen-Macaulay ring and $r_d(R) = 1$. By Corollary 3.6, the index of reducibility for all distinguished parameter ideals of R are eventually 1.

If. Since the index of reducibility for all distinguished parameter ideals of R are eventually 1 there exists an integer n such that for all distinguished parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^n$, we have

$$\mathcal{N}(\mathfrak{q}; R) = 1 \leq r_d(R).$$

By the Corollary 3.6, R is Cohen-Macaulay and so $r_d(R) = 1$. Hence R is Gorenstein, as required.

□

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